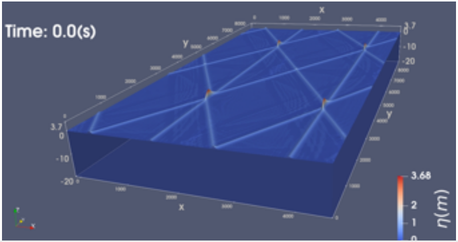
Extreme events in Wetropolis flood investigator & dynamics of extreme water waves

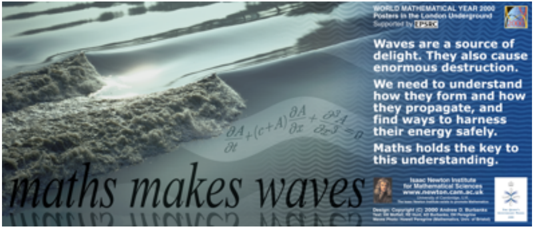
Onno Bokhove [et al.], KAIST 21-08-2024

£€: EU Eagre GA859983

Leeds Institute for Fluid Dynamics, UK



Discussion on Wetropolis World and Aspects of wave-energy device



(Painting V. Zwart)

Discussion on Wetropolis World

Question: Is it unusual for a mathematician to build or propose fluid-dynamical devices and demonstrations? 몰라요:

- The inventor of the Galton board "Sir Francis Galton was a British poly-math ..." (and mathematician).
- The innovation of Wetropolis lies in the coupling between the weather or rain machine with its skew-Galton boards and the conceptual river catchment.
- Underlying Wetropolis is a mathematical and numerical design model of PDEs, ODEs and diagnostic relations linking the equations for various components.
- Wetropolis is one member of suite of fluid-dynamical demonstrations created with designer Wout Zweers.

Is it unusual for a mathematician to propose fluid-dynamical devices?

A question from a KAIST member on 15-06-2024. 몰라요:

- Wetropolis is one member of suite of fluid-dynamical demonstrations, often based on mathematical and numerical design models.
- Note that a design model aims to accommodate a design and is generally not a suitable or detailed predictive model.

A suite of mathematical demonstration devices 2010-2024

- 2010 Beach formation in a vertical Hele-Shaw cell: beach formation of one lateral layer of particles by breaking waves. Design model on wave breaking without beach or moving bottom dynamics determined the gap width. Hele-Shaw: "Hele Shaw was an English mechanical and automobile engineer ... Various problems in fluid dynamics can be approximated to Hele-Shaw flows". Portable version at ICCE2014 in Seoul https://www.youtube.com/watch?v=iz_S-NlLYyU
- ...: Built in honour of the late Prof Howell Peregrine (mathematician) for an art-maths show Fluid Fascinations using Peregrine's slides of fluid-dynamics phenomena inherited via School of Maths, Bristol, UK. Series of papers.

. . . mathematical demonstration devices 2010-2024

- 2010 Bore-soliton splash: no a-priori design model but a small-scale trial experiment. Inspired by Peregrine's work on wave impact against vertical seawalls and my work on hydraulic and granular flows through contractions. Series of papers. https://www.youtube.com/watch?v=YSXsXNX4zW0
- 2013 Wave-energy device: proof of principle. No a-priori design but test to see if subsequent mathematical model worthwhile formulating. Series of papers, also at European Wave and Tidal Energy conferences.
- 2015 Coastal wave tank: MSc team project and commission by JBA Trust. Centre for Doctoral Training (CDT) in Fluid Dynamics, Leeds. 10M views on YouTube and 100M views on Fossbytes. Modelling and design done in unison.

https://www.youtube.com/watch?v=3yNoy4H2Z-o and design

. . . mathematical demonstration devices 2010-2024

- 2016-now Wetropolis flood investigator. Request by Environment Agency and JBA Trust to visualise return periods in a physical and portable set-up. Has design model.
- . . .
- 2023 Wet canopy evaporation: MSc CDT team project on water evaporation by forest/trees and the flood-mitigation effects this may have. Limited, probing design model.
- GitHub on some of these designs:

https://github.com/obokhove/MathslaboratoryUoL

Wetropolis design model

... white/blackboard `https://github.com/tkent198/hydraulic_wetro`:

$$\text{River}: \partial_t (w_r h) + \partial_s \left(w_r h R(h)^{2/3} \sqrt{-\partial_s b}/C_m \right) = Q_m \delta(s - s_m) + Q_{res} \delta(s - s_{res}) + Q_{1c} \delta(s - L_{1c}) \text{ on } s \in [0, L],$$
$$\text{with } Q_f|_{s=0} = w_r h R(h)^{2/3} \sqrt{-\partial_s b}/C_m|_{s=0} = Q_0(t),\ h(s,0) = h_0(s). \tag{14a}$$

$$\text{Moor}: \partial_t (w_v h_m) - \alpha g \partial_y \left(w_v h_m \partial_y h_m \right) = \frac{w_v R_m(t)}{m_{por} \sigma_e},$$
$$\text{on } y \in [0, L_y],\ \text{with } \partial_y h_m|_{y=L_y} = 0,\ h_m(0,t) = h_{3c}(t),\ h_m(y,0) = h_{m_0}(y). \tag{14b}$$

$$\text{Reservoir}: w_{res} L_{res} \frac{dh_{res}}{dt} = w_{res} L_{res} R_{res}(t) - Q_{res},\ \text{with } h_{res}(0) = h_{res0}. \tag{14c}$$

$$\text{Canal-1}: w_c (L_{1c} - L_{2c}) \frac{dh_{1c}}{dt} = Q_{2c} - Q_{1c},\ \text{with } h_{1c}(0) = h_{10}. \tag{14d}$$

$$\text{Canal-2}: w_c (L_{2c} - L_{3c}) \frac{dh_{2c}}{dt} = Q_{3c} - Q_{2c},\ \text{with } h_{2c}(0) = h_{20}. \tag{14e}$$

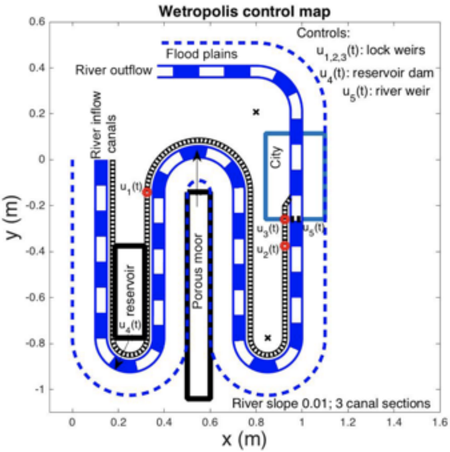
$$\text{Canal-3}: w_c L_{3c} \frac{dh_{3c}}{dt} = \gamma Q_{tm} - Q_{3c},\ \text{with } h_{3c}(0) = h_{30}. \tag{14f}$$

with

$$Q_{1c} = C_f \sqrt{g} w_c \max(h_{1c} - P_{1w}, 0)^{3/2},\quad Q_{2c} = C_f \sqrt{g} w_c \max(h_{2c} - P_{2w}, 0)^{3/2},\quad Q_{3c} = C_f \sqrt{g} w_c \max(h_{3c} - P_{3w}, 0)^{3/2}, \tag{14g}$$

$$Q_m = (1 - \gamma) Q_{tm} \equiv (1 - \gamma) \frac{1}{2} m_{por} \sigma_e w_v \alpha g \left(\partial_y h_m \right)^2 |_{y=0}, \tag{14h}$$

$$Q_{res} = C_f \sqrt{g} w_{res} \max(h_{res} - P_{wr}, 0)^{3/2} \text{ and } R(h) = w_r h / (2h + w_r). \tag{14i}$$



Wetropolis World discussion

Principle objectives:

- To use the reduced (relative to NWP) physical and modelling Wetropolis World to study pros and cons of classical (P)DEs and data-assimilation based flood predictions with ones arising from ML. Goal: "**Numerical Wetropolis Prediction**".
- To assess **info-gap theory** to hind- and forecast catchment- and small-scale flood-mitigation plans.
- Info-gap theory assesses the robustness of decisions against lacking information (the "info-gap") and uncertainty, often leading to different decisions relative to optimisation against only known (uncertain) information.
- See High Beck fluvial case study at European Geophysical Assembly 2024 in which unvalued co-benefits of flood-mitigation measures assessed against threshold values.

. . . Wetropolis World discussion

- **Serious games:** To design saleable Wetropolis board game with two 16-faced dice replacing the weather machine, flood wave out of the board, sewage overflow, fragility of break-down risk (another dice throw) of flood defenses, and flood-control options (gate height at reservoirs).
- **Serious games:** To arrange saleable reduced 1/4-size industry-proof version of Wetropolis for institutes, weather centres, schools and universities.
- To promote **user-friendly decision-making tools** on flood-mitigation measures.
- Collaborations and adaptations are welcome!

Aspects of wave-energy device: results to date

To date the following modelling and results have been obtained:

- Full nonlinear wave-to-wire model formulated for both equality and inequality constraints of the 2D buoy-free-surface water interface (B et al 2019).
- Full numerical 2D linearised (shallow-water) model of wave-to-wire model with equality constraint (B. et al 2019, Bolton et al. 2021, B et al IEEE2024).
- *Optimisations* done for the contraction geometry using full and (Latin-hypercube) surrogate modelling (B et al IEEE2024).

Aspects of wave-energy device: results to date

...following modelling and results have been obtained:

- Full nonlinear 3D potential-flow water-wave submodel & numerical simulations (Gidel et al 2022, Choi et al 2022/2024, Lu et al 2024), implementing model's space-time variational principle (VP). 2nd order in time, higher-order in space.
- Full nonlinear buoy-generator submodel & numerics based on time-discrete VP plus symmetric/consistent dissipative terms.
- Optimisation of (linear) buoy-generator submodel by analysing one-coil and three-coils-in-parallel generators with inductance $L_I = 3L_I^{(i)}$.

Why variational principles (VPs)? Advantages:

- When (main) dynamics has a VP, multiple coupled equations are succintly described by one space-time VP.
- Associated with a VP are conservation properties of associated PDEs/ODEs.
- Within the *(finite-element) environment Firedrake*, the time-discrete VP can be implemented directly, with automated generation of (complicated 3D+1D) weak forms of the equations. Adjoint solvers, optimal control.
- Advantages: enormous reduction in development time, efficient, flexible, higher-order spectrally-accurate space discretisations plus (automatic) preservation of discrete forms of conservation properties.

Numerical analysis of inequality constraints

Strategy on buoy-water surface coupling (work in progress):

- Goal is to solve Bernoulli & KKT inequality equations:

$$\delta h: \quad \partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + g(z - H_0) - \lambda = 0 \quad \text{at} \quad z = h.$$
$$\delta \lambda: \lambda = -[\gamma(h - h_b) - \lambda]_+ = -F_+(\gamma(h - h_b) - \lambda)$$
$$\Longrightarrow \underline{h(x,t) - h_b(Z,x) \leq 0}, \lambda \leq 0, \lambda(h - h_b) = 0.$$

- Analyse rest-flow case, when $\phi = 0$ with $h_b(Z,x) = Z - K - \tan\theta(L_y - x)$.
- Analyse "simpler" problem of a ball with position $Z(t)$ falling under gravity to a flat surface such that $Z(t) \geq 0$. **Done.**
- Use Firedrake's in-built inequality-constraint solvers.

Vertically-falling ball under gravity with $Z(t) \geq 0$

Continuous in time:

- VP of falling ball with unit mass $M = 1$ without constraint:

$$0 = \delta\mathcal{F} = \delta \int_0^T L(Z, W)\, \mathrm{d}t \equiv \delta \int_0^T W\dot{Z} - \frac{1}{2}W^2 - Z\, \mathrm{d}t,$$
$$\equiv \lim_{\epsilon \to 0} \int_0^T \frac{L(Z + \epsilon\delta Z, W + \epsilon\delta W) - L(Z, W)}{\epsilon}\, \mathrm{d}t$$

  time t, acceleration $g = 1$, kinetic & potential energy MgZ.
- Minimisation problem with *virtual* changes $z_p = \delta Z$ and $w_p = \delta W$, i.e. $\delta Z(0) = \delta Z(T) = 0$.
- Newton's equations for position $Z(t)$ and velocity $W(t) \equiv \dot{Z}$:

$$0 = \int_0^T (\dot{Z} - W)\delta W - (\dot{W} + 1)\delta Z\, \mathrm{d}t : \dot{Z} = W, \quad \dot{W} = -1.$$

Vertically-falling ball under gravity with $Z(t) \geq 0$

Continuous in time:

- VP of falling ball with inequality constraint:

$$
\begin{aligned}
0 =& \delta \int_0^T L(Z, W)\,\mathrm{d}t \\
\equiv& \delta \int_0^T W\dot{Z} - \frac{1}{2}W^2 - Z - \frac{1}{2\gamma}\left(F_+(-\gamma Z - \lambda)^2 - \lambda^2\right)\,\mathrm{d}t,
\end{aligned}
$$

- Resulting equations:

$$
\begin{aligned}
\delta W: \quad & \dot{Z} = W \\
\delta Z: \quad & \dot{W} = -1 - \lambda \\
\delta \lambda: \quad & \lambda = -F_+(-\gamma Z - \lambda)F'_+(-\gamma Z - \lambda) \\
& \quad = -F_+(-\gamma Z - \lambda) \iff \underline{-Z \leq 0}, \lambda \leq 0, \lambda Z = 0.
\end{aligned}
$$

Vertically-falling ball: smoothing

Approximations of F_+ include:

$$F_+(q) = \frac{1}{2}q + \sqrt{b^2 + \frac{1}{4}q^2} \rightarrow_{b\rightarrow 0} \max(q,0) \quad \text{with}$$
$$F'_+(q) = \frac{1}{2} + \frac{\frac{1}{4}q}{\sqrt{b^2 + \frac{1}{4}q^2}} \rightarrow_{b\rightarrow 0} \Theta(q)$$

for $b > 0$.

Vertically-falling ball: smoothing

Hence,

$$\lambda = -F_{+}(-\gamma Z - \lambda) = -\left(-\frac{1}{2}(\gamma Z + \lambda) + \sqrt{b^2 + (\gamma Z + \lambda)^2/4}\right)$$

$$\iff \frac{1}{2}(\lambda - \gamma Z) = -\sqrt{b^2 + (\gamma Z + \lambda)^2/4} \Longrightarrow$$

$$-\gamma Z \lambda = b^2 \quad \text{for} \quad Z \geq 0 \iff$$

$$\lambda = -\frac{b^2}{\gamma Z} \quad \text{for} \quad Z \geq 0.$$

Vertically-falling ball: phase plot

Therefore, equations become as follows and can be solved

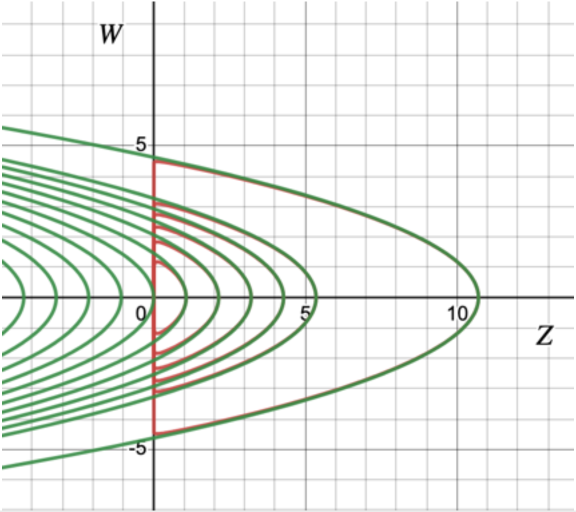
$$\dot{Z} = W, \dot{W} = -1 + \frac{b^2}{\gamma Z} \Longleftrightarrow \ddot{Z} = -1 + \frac{b^2}{\gamma Z} \Longrightarrow$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}W^2 + Z - \frac{b^2}{\gamma}\ln Z\right) = 0 \Longleftrightarrow$$

$$\frac{1}{2}W^2 + Z - \frac{b^2}{\gamma}\ln Z = H_0$$

with integration constant/energy $H_0 = H(t)$. When $W = 0$ maximum $Z = Z_{max}$ satisfies $Z_{max} - b^2/(\gamma Z_{max}) = H_0$ with as first approximation $Z_{max} \approx H_0 = H_1 - b^2/(\gamma) \ln H_1$, where $\frac{1}{2}W^2 + Z = H_1$. Make phase plot in (Z, W)-plane:

Vertically-falling ball: phase plot



Vertically-falling ball under gravity with $Z(t) \geq 0$

Time discrete VP:

- 2nd-order modified mid-point VP of falling ball with constraint:

$$0 = \delta\Big(W^{n+1/2}\frac{Z^{n+1}-Z^n}{\Delta t} - Z^{n+1/2}\frac{W^{n+1}-W^n}{\Delta t} \\ - \frac{1}{2}(W^{n+1/2})^2 - Z^{n+1/2} - \frac{1}{2\gamma}\Big(F_+(-\gamma Z^{n+1/2}-\lambda)^2 - \lambda^2\Big)\Big)$$

with additional relations

$$Z^{n+1} = 2Z^{n+1/2} - Z^n \text{ and } W^{n+1} = 2W^{n+1/2} - W^n.$$

Vertically-falling ball under gravity with $Z(t) \geq 0$

- Resulting time-discrete equations (variations wrt $Z^{n+1/2}, W^{n+1/2}$):

$$\delta W^{n+1/2}: \; Z^{n+1} = Z^n + \Delta t W^{n+1/2} \Longrightarrow$$

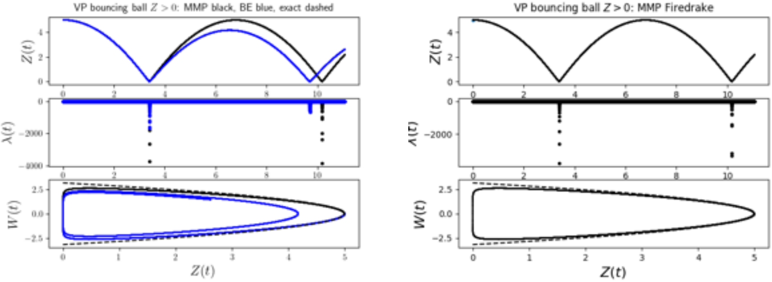
$$Z^{n+1/2} = Z^n + \frac{1}{2}\Delta t W^{n+1/2}$$

$$\delta Z^{n+1/2}: \; W^{n+1} = W^n - \Delta t(1+\lambda)$$

$$\Longrightarrow \frac{4(Z^{n+1/2} - Z^n)}{\Delta t} = 2W^n - \Delta t(1+\lambda)$$

$$\delta\lambda: \; \lambda = -F_+(-\gamma Z^{n+1/2} - \lambda)F_+'(-\gamma Z^{n+1/2} - \lambda).$$

Vertically-falling ball under gravity with $Z(t) \geq 0$



Concluding remarks

- Rest-flow case as stepping stone is in progress.
- For the dissipative terms in the electro-magnetic generator, either a symmetric Crank-Nicolson discretisation or integrating factor is required: backward, forward & sympletic (Euler) discretisations are unstable/wrong.
- A laboratory set-up of the wave-energy device is under construction, in steps:
  a) dry & driven buoy-generator hanging from a spring, and
  b) full hydrodynamic coupling,
  in our $2\mathrm{m}$ wave tank.

Thanks very much for your attention ...

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